Conformally flat Einstein-like 4-manifolds and conformally flat Riemannian 4-manifolds all of whose Jacobi operators have parallel eigenspaces along every geodesic

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Abstract

A local classification of all locally conformal flat Riemannian 4-manifolds whose Ricci tensor satisfies the equation $\nabla\left(\rho - \frac{1}{6}sg\right) = \frac{1}{18}ds \odot g$ as well as a local classification of all locally conformal flat Riemannian 4-manifolds for which all Jacobi operators have parallel eigenspaces along every geodesic is given. Non-trivial explicit examples are presented. The problem of local description of self-dual Einstein-like 4-manifolds is also treated. A complete explicit solution of the Stäckel system in dimension 4 is obtained.

Running title: Einstein-like 4-manifolds

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1 Introduction

One of the most important Riemannian metric on a smooth manifold is the Einstein metric. In 1978 A.Gray [17] suggested to study three generalizations of Einstein metrics. Background for his investigations was the fact that any Einstein metric has parallel Ricci tensor, and conversely, that any Riemannian manifold with parallel Ricci tensor is locally a product of Einstein metrics. Using representation theory of the orthogonal group Gray decomposed the covariant derivative of the Ricci tensor of a Riemannian manifold into its irreducible components and derive so in a natural way three classes of Riemannian manifolds, namely: the class $\mathcal{B}$ of Riemannian manifolds whose Ricci tensor is a Codazzi tensor (this is precisely the class of Riemannian manifolds with harmonic curvature), the class $\mathcal{U}$ of Riemannian manifold with Killing Ricci tensor and the class $\mathcal{Q}$ of Riemannian manifolds whose Ricci tensor satisfies

$$\nabla\left(\rho - \frac{1}{2n-2}sg\right) = \frac{n-2}{2(n+2)(n-1)}ds \odot g \tag{1.1}$$

where n is the dimension of the manifold, g its Riemannian metric, ∇ the Levi-Civita connection of the metric g, ρ the Ricci tensor, s the scalar curvature and $\odot$ denotes the symmetric product of symmetric tensors.

In this paper we concentrate our attention on $\mathcal{Q}$-spaces (for the sake of brevity we call manifolds belonging to the class $\mathcal{Q}$ also $\mathcal{Q}$-spaces).

A complete local description (up to an isometry) of the 3-dimensional $\mathcal{Q}$-spaces is given by J. Berndt in [1]. But, in higher dimensions only few examples for $\mathcal{Q}$-spaces with non-parallel Ricci tensor are known. A class of examples for $\mathcal{Q}$-spaces was presented in [7], p.448. These examples arise as certain bundles over 1-dimensional manifolds whose fiber is an Einstein manifold with negative scalar curvature. If the fiber is a space of constant sectional curvature then these examples are locally conformal flat $\mathcal{Q}$-spaces. Besse considered also the problem of local description of $\mathcal{Q}$-spaces, but he could not obtain any classification of these spaces. As we know, this problem is still open.

In the paper we give a particular answer to this classification problem. We describe locally (up to an isometry) the 4-dimensional locally conformal flat $\mathcal{Q}$-spaces. Our considerations based on the fact that for every locally conformal flat $\mathcal{Q}$-space the Jacobi operator and its covariant derivative commute. Riemannian manifolds for which the Jacobi operator has this property are introduced by J.Berndt and L. Vanhecke in [4] as a natural generalization of locally symmetric Riemannian spaces. These spaces are called $\mathcal{P}$-spaces in [4]. A $\mathcal{P}$-space is by definition a Riemannian manifold all of whose Jacobi operators have parallel eigenspaces along every geodesic. The $\mathcal{P}$-spaces are also of special interest (see [5, 6, 2, 3]). A complete local classification of the 2-dimensional and the 3-dimensional $\mathcal{P}$-spaces is given in [4]. In higher dimensions however, few examples for not locally symmetric $\mathcal{P}$-spaces are known. As we know, the problem of local classification of $\mathcal{P}$-spaces is not yet solved in higher dimensions.

In this note we classify locally the 4-dimensional locally conformal flat $\mathcal{Q}$-spaces and locally conformal flat $\mathcal{P}$-spaces. To this end, we write down 10 kinds of explicit Riemannian metrics on $\mathbf{R^4}$. Following the ideas of L.P.Eisenhart in [13] we show that these metrics give a complete solution of the so called Stäckel system in dimension 4. The present classification provides also some new examples of $\mathcal{Q}$-spaces and some new examples of

$\mathcal{P}$-spaces which seem to be unknown up to now. The purpose of this note is to prove the following two local structure theorems

Theorem 1.1 *Let $(\mathbf{M}, g)$ be a connected 4-dimensional locally conformal flat $\mathcal{P}$-space. Then $(\mathbf{M}, g)$ is locally (almost everywhere) isometric to one of the following spaces*

I) a real space form;

II) a Riemannian product of two 2-dimensional real space forms of opposite constant sectional curvatures;

III_1) a warped product $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$, where $\mathbf{B^1}$ is a 1-dimensional space, $\mathbf{N^3}$ is a 3-dimensional real space form and $\mathbf{f}$ is a positive smooth function on $\mathbf{B^1}$.

IV) a warped product $\mathbf{B^2} \times_{\mathbf{f}} \mathbf{N^2}$, where $\mathbf{N^2}$ is a 2-dimensional real space form of constant sectional curvature $\mathcal{K}_{\mathrm{N}}$, $\mathbf{B^2}$ is (a domain of) $\mathbf{R^2}$ with the following Riemannian metric

$$g = \beta(x)\gamma(x)dx^2 + \left(\left(\beta(x)y + \alpha(x)\right)dx + dy\right)^2,$$

where

$$\beta(x) = -\frac{\mathcal{K}'(x)}{2\left(\mathcal{K}(x) + \frac{\mathcal{K}_{\mathrm{N}}}{cA} + A\right)}, \quad \gamma(x) = \frac{\mathcal{K}'(x)}{\mathcal{K}^2(x) - \left(\frac{\mathcal{K}_{\mathrm{N}}}{cA}\right)^2},$$

$\alpha(x)$, $\mathcal{K}(x)$ are smooth functions of x, $\mathcal{K}'(x) \neq 0$, c, A are constants, different from zero, such that $\mathcal{K}(x) \neq \pm\frac{\mathcal{K}_{\mathrm{N}}}{cA}$, $\mathcal{K}(x) \neq -\frac{\mathcal{K}_{\mathrm{N}}}{cA} - A$, $c\mathcal{K}(x) + \frac{\mathcal{K}_{\mathrm{N}}}{A} > 0$ for every x. The positive function f is determined by

$$\mathrm{f}^2(x) = c\mathcal{K}(x) + \frac{\mathcal{K}_{\mathrm{N}}}{A}.$$

The Gauss curvature of $\mathbf{B^2}$ is equal to $\mathcal{K}$;

V) a warped product $\mathbf{B^2} \times_{\mathbf{f}} \mathbf{N^2}$, where $\mathbf{N^2}$ is a 2-dimensional real space form of constant sectional curvature $\mathcal{K}_{\mathrm{N}}$, $\mathbf{B^2}$ is (a domain of) $\mathbf{R^2}$ with the following Riemannian metric

$$g = E(x)\left(\mu\left(x,y\right) + D\left(x\right)\right)dx^2 + \left(\frac{\mu_x(x,y)}{\mu_y(x,y)}dx + dy\right)^2,$$

where

$$E(x) = \frac{(D'(x))^2}{2D^2(x)\left((D(x) - C)^2 + \frac{2\mathcal{K}_{\mathrm{N}}}{cD(x)} + e\right)},$$

$D(x)$ is a smooth function of x, $D(x) \neq 0$, $D'(x) \neq 0$ for every x, C, c, e are constants, $c \neq 0$ and $\mu(x,y)$ is a smooth solution of the following PDE

$$\mu_y^2(x,y) = \frac{2\mu(x,y)}{\mu(x,y) + D(x)}\left[\mu(x,y)\left((\mu(x,y) + C)^2 + e\right) - \frac{2\mathcal{K}_{\mathrm{N}}}{c}\right]$$

such that $\mu_y(x,y) \neq 0$, $cD(x)\mu(x,y) > 0$, $E(x)(\mu(x,y) + D(x)) > 0$ for every (x,y). The positive function f is determined by

$$\mathrm{f}^2(x,y) = cD(x)\mu(x,y).$$

The Gauss curvature $\mathcal{K}$ of $\mathbf{B^2}$ is equal to $\mathcal{K} = \mu - D + C$;

VI) a Riemannian manifold with the Riemannian metric of the form

$$g = dx_1^2 + \phi^2(x_1)dx_2^2 + r\left(\phi^2\left(x_1\right) + a\right)dx_3^2 + q\left(\phi^2\left(x_1\right) + b\right)dx_4^2,$$

where a, b, q, r are positive constants, $a \neq b$ and $\phi(x)$ is a non-constant solution of the following ordinary differential equation

$$(\phi'(x))^2 = \phi^4(x) + (a+b)\phi^2(x) + ab.$$

VII) a Riemannian manifold with the Riemannian metric of the form

$$g = \sum^{\sigma}_{1,2,3,4} F_1(x_1)|x_1 - x_2||x_1 - x_3||x_1 - x_4|dx_1^2,$$

where σ denotes the cyclic sum and $1/F_i(x_i) = P(x_i)(i = 1,2,3,4)$ with a polynomial $P(x)$ of degree six.

VIII) a warped product $\mathbf{B^3} \times_{\mathbf{f}} \mathbf{N^1}$, where $\mathbf{N^1}$ is a one-dimensional Riemannian manifold, $\mathbf{B^3}$ is a three-dimensional Riemannian manifold with Riemannian metric of the form

$$g = \sum^{\sigma}_{2,3,4} F_2(x_2)|x_2 - x_3||x_2 - x_4|dx_2^2,$$

where σ denotes the cyclic sum and $1/F_i(x_i) = P(x_i), (i = 2,3,4)$ with a polynomial $P(x)$ of degree five having zero as a root. The function f is given by

$$f^2 = |x_2 x_3 x_4|.$$

IX) a Riemannian manifold with Riemannian metric of the form

$$g = (x_3 - b)(x_4 - b)dx_1^2 + x_3 x_4 dx_2^2 + \sum^{\sigma}_{3,4} F_3(x_3)|x_3 - x_4|dx_2^2,$$

where σ denotes the cyclic sum and $1/F_i(x_i) = P(x_i)(i = 3,4)$ with a polynomial $P(x)$ of degree four given by $P(x) = (x-b)(a_3x^3 + a_2x^2 + a_1x), \quad b \neq 0.$

Conversely, every Riemannian manifold of type I), II), III_1), IV), V), VI), VII), VIII) and IX) is a locally conformal flat $\mathcal{P}$-space.

Theorem 1.2 *Let $(\mathbf{M}, g)$ be a connected 4-dimensional locally conformal flat $\mathcal{Q}$-space. Then $(\mathbf{M}, g)$ is a $\mathcal{P}$-space and it is locally (almost everywhere) isometric to one of the spaces I), II), IV), V), VI), VII), VIII), IX) or to*

III_2) a warped product $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ of a 1-dimensional base $\mathbf{B^1}$ and a 3-dimensional leaf $\mathbf{N^3}$ with constant sectional curvature $\mathcal{K}_{\mathrm{N}}$, and $F = 1/\mathrm{f}$ is a positive solution of the following second order differential equation

$$(\mathrm{F}''(x))^2 = 2\mathcal{K}_{\mathrm{N}}\mathrm{F}^3 + c\mathrm{F}, \tag{1.2}$$

where c is a constant.

Conversely, every Riemannian manifold of type I), II), III_2), IV), V), VI), VII), VIII) and IX) is a locally conformal flat $\mathcal{Q}$-space.

If the Jacobi operator has pointwise constant eigenvalues then the Riemannian manifold is said to be a pointwise Osserman manifold. By the result of P.Gilkey, A.Swann and L.Vanhecke in [16], a 4-manifold is pointwise Osserman manifold iff locally there exists a choice of orientation such that the manifold is self-dual and Einstein. It is well known (see e.g. [7], 16.4) that every 4-dimensional $\mathcal{Q}$-space has harmonic Weyl tensor. The results of J.P.Bourguignon [9], A.Derdzinski [11], A.Derdzinski and C.L.Chen [12] (see also [7], 16.29) imply that any self-dual or anti-self-dual 4-dimensional $\mathcal{Q}$-space is either with zero Weyl tensor or Einstein. Combining these results with Theorem 1.2 we obtain

Corollary 1.3 *Let* $(\mathbf{M},g)$ *be a connected 4-dimensional self-dual (or anti-self-dual)* $\mathcal{Q}$*-space. Then* $(\mathbf{M},g)$ *is locally (almost everywhere) either Einstein (and hence pointwise Osserman) or isometric to one of the spaces described in Theorem 1.2.*

Thus, the description of self-dual 4-dimensional $\mathcal{Q}$-space is reduced to the difficult problem of local classification of pointwise Osserman 4-manifolds which is equivalent to the local description of self-dual Einstein 4-manifolds (see the recent work of N.Hitchin [19] for the latter problem).

Remarks. **1.** The spaces described in III_2) and VII) of Theorem 1.2 are the locally conformal flat $\mathcal{Q}$-spaces presented in [7].

2. The spaces described in III_1) and III_2) have parallel Ricci tensor iff the function f is either constant or f is determined by one of the following three conditions:

a) $\mathrm{f} = \varepsilon\sqrt{\mathcal{K}_{\mathrm{N}}}x + b$, where $\varepsilon = \pm 1$, $b = const$, $\mathcal{K}_{\mathrm{N}} > 0$,

b) $\mathrm{f} = Ce^{ax} + De^{-ax}$, where a, C, D are constants, $\mathcal{K}_{\mathrm{N}} + 4CDa^2 = 0$,

c) $\mathrm{f} = C\sin(ax) + D\cos(ax)$, where a, C, D are constants and $\mathcal{K}_{\mathrm{N}} = a^2(C^2 + D^2)$ if $\mathcal{K}_{\mathrm{N}} > 0$.

3. The spaces of type VI) are always with non-parallel Ricci tensor.

4. The spaces of types VII), VIII) and IX) are also with non-parallel Ricci tensor. If we consider the polynomial P(x) of degree less then six in the case VII), P(x) of degree less then five in the case VIII), P(x) of degree less then four in the case IX), then the Ricci tensor is parallel, but this leads us to spaces of constant curvature.

5. Comparing Theorem 1.1 and Theorem 1 of [10] we can conclude that every Riemannian 4-manifold with harmonic curvature whose Ricci tensor has at most two distinct eigenvalues is a $\mathcal{P}$-space.

6. A locally conformal flat Riemannian manifold has constant Ricci eigenvalues, i.e. it is curvature homogeneous, iff it is locally symmetric Riemannian manifold [26],[20]. Thus, we obtain by Theorem 1.1 and Theorem 1.2 examples of locally conformal flat $\mathcal{Q}$-spaces and examples of locally conformal flat $\mathcal{P}$-spaces which are not even curvature homogeneous since locally conformal flat curvature homogeneous Riemannian 4-manifolds are exactly the spaces with constant Ricci eigenvalues. All these examples are not also pointwise Osserman manifolds by Corollary 1.3.

2 Preliminaries

Let $(\mathbf{M}, g)$ be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of the metric g. The curvature R of ∇ is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for every vector fields X, Y on $\mathbf{M}$. We denote by $T_p\mathbf{M}$ the tangential space at a point $p \in \mathbf{M}$.

Let $X \in T_p\mathbf{M}$. The Jacobi operator is defined by

$$\lambda_X(Y) = R(Y,X)X, \qquad Y \in T_p\mathbf{M}.$$

Let $\gamma(t)$ be a geodesic on $\mathbf{M}$ and $\dot\gamma$ denotes its tangent vector field. We consider the family of smooth self-adjoint Jacobi operators along γ defined by $\lambda_{\dot\gamma} = R(X,\dot\gamma)\dot\gamma$ for every smooth vector field X along γ. A Riemannian manifold $(\mathbf{M}, g)$ is said to be a $\mathcal{P}$-space if the operators $\lambda_{\dot\gamma}$ have parallel eigenspaces along every geodesic on $\mathbf{M}$ [4].

A Riemannian manifold $(\mathbf{M}, g)$ is said to be locally conformal flat if around every point $p \in \mathbf{M}$ there exists a metric $\bar{g}$ which is conformal to g and $\bar{g}$ is flat. By the Weyl theorem, an n-dimensional $(n \geq 4)$ Riemannian manifold is locally conformal flat iff the Weyl curvature tensor W vanishes (see e.g. [24]) i.e. the curvature of the metric g has the following form

$$R(X,Y,Z,U) = -\frac{s}{(n-1)(n-2)}\left(g\left(Y,Z\right)g(X,U) - g(X,Z)g(Y,U)\right) + \tag{2.3}$$

$$+\frac{1}{n-2}\left(\rho\left(Y,Z\right)g(X,U) - \rho(X,Z)g(Y,U) + g(Y,Z)\rho(X,U) - g(X,Z)\rho(Y,U)\right),$$

where $X, Y, Z, U \in T_p\mathbf{M}$, $p \in \mathbf{M}$. The following condition also holds

$$(\nabla_X\rho)(Y,Z) - (\nabla_Y\rho)(X,Z) = \frac{1}{2(n-1)}\left(X\left(s\right)g(Y,Z) - Y(s)g(X,Z)\right). \tag{2.4}$$

3 Proof of Theorem 1.1

The Ricci operator Ric is defined by $g(Ric(X), Y) = \rho(X, Y), X, Y \in T_p\mathbf{M}, p \in \mathbf{M}$. In every point $p \in \mathbf{M}$ we consider the Ricci operator Ric as a linear self-adjoint operator on the tangential space $T_p\mathbf{M}$. Let Ω be the subset of $\mathbf{M}$ on which the number of distinct eigenvalues of Ric is locally constant. This set is open and dense on $\mathbf{M}$. We can choose C^∞ eigenvalue functions of Ric on Ω, say r_1, r_2, r_3, r_4, in such a way such that they form the spectrum of Ric at each point of Ω (see for example [22, 25, 4]). We fix a point $p \in \Omega$. Then there exists a local orthonormal frame field E_1, E_2, E_3, E_4 on an open connected neighborhood V of p such that

$$Ric(E_i) = r_iE_i, \qquad i = 1, 2, 3, 4.$$

There exists an open connected neighborhood $U_p \subset V$ of p such that we have either $r_i = r_j$ or $r_i \neq r_j$ everywhere on U_p, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$

We fix the neighborhood U_p and set

$$(\nabla_i\rho)_{jk} = (\nabla_{E_i}\rho)(E_j, E_k), \quad \omega_{ij}^k = g(\nabla_{E_i}E_j, E_k), \quad i, j, k \in \{1, 2, 3, 4\}.$$

We have

$$(\nabla_i\rho)_{jk} = \delta_{jk}E_i(r_i) + (r_j - r_k)\omega_{ij}^k, \quad i, j, k \in \{1, 2, 3, 4\}, \tag{3.5}$$

where δ_{jk} is the Kroneker's symbol.

Further, unless otherwise stated, the Latin indices i, j, k, l will stand for any distinct integers from the set $\{1, 2, 3, 4\}$.

We need the following technical result

Proposition 3.1 *Four-dimensional locally conformal flat Riemannian manifold $(\mathbf{M}, g)$ is a $\mathcal{P}$-space iff for every point $p \in \Omega$ the following conditions hold on U_p*

$$(\nabla_i\rho)_{jk} = 0, \tag{3.6}$$

$$(\nabla_j\rho)_{jk} = (\nabla_i\rho)_{ik}, \tag{3.7}$$

$$E_i(r_k) = E_i(r_j), \tag{3.8}$$

$$E_i(r_i) = 3\left(E_i(r_k) - 2(\nabla_k\rho)_{ki}\right), \tag{3.9}$$

$$(r_i - r_l)\left(E_i(r_i) - E_i(r_k) - 2(\nabla_k\rho)_{ki}\right) = 0. \tag{3.10}$$

Proof of Proposition 3.1. Let $X \in T_p\mathbf{M}$, $p \in \mathbf{M}$. We consider the following skew symmetric operator L_X on $T_p\mathbf{M}$ defined by $L_X = \lambda'_X \circ \lambda_X - \lambda_X \circ \lambda'_X$, where λ'_X is the covariant derivative of the Jacobi operator defined by $\lambda'_X(Y) = (\nabla_X R)(Y, X, X), Y \in T_p\mathbf{M}$. We know from [4] that $(\mathbf{M}, g)$ is a $\mathcal{P}$-space iff the Jacobi operator and its covariant derivative commute, i.e. $L_X = 0$. Because of (2.3) and (2.4), the latter equality is equivalent to

$$(\rho(Y,Y) - \rho(Z,Z))\,(\nabla_X\rho)(Y,Z) + ((\nabla_X\rho)(Z,Z) - (\nabla_X\rho)(Y,Y))\,\rho(Y,Z)+ \tag{3.11}$$

$$+\rho(Y,U)(\nabla_X\rho)(Z,U) - \rho(Z,U)(\nabla_X\rho)(Y,U) = 0,$$

where X, Y, Z, U are orthonormal vectors of $T_p\mathbf{M}$, $p \in \mathbf{M}$.

Let $(\mathbf{M}, g)$ be a $\mathcal{P}$-space. Taking $X = E_i$, $Y = E_j$, $Z = E_k$, $U = E_l$, we obtain from (3.11) that

$$(r_j - r_k)(\nabla_i\rho)_{jk} = 0.$$

The latter equality and (3.5) imply (3.6). Setting $X = \cos\alpha E_i + \sin\alpha E_j$, $\quad Y = \sin\alpha E_i - \cos\alpha E_j$, $\quad Z = \cos\beta E_k + \sin\beta E_l$, $\quad U = \sin\beta E_k - \cos\beta E_l$ into (3.11), we obtain

$$\sin 2\alpha \left\{2\left(r_i \sin^2\alpha + r_j\cos^2\alpha - r_k\cos^2\beta - r_l\sin^2\beta\right)\,. \right. \tag{3.12}$$

$$(\cos\beta\,((\nabla_i\rho)_{ik} - (\nabla_j\rho)_{jk}) + \sin\beta\,((\nabla_i\rho)_{il} - (\nabla_j\rho)_{jl})) -$$

$$\left. -(r_k - r_l)\sin 2\beta\,(\sin\beta\,((\nabla_i\rho)_{ik} - (\nabla_j\rho)_{jk}) - \cos\beta\,((\nabla_i\rho)_{il} - (\nabla_j\rho)_{jl}))\right\} = 0.$$

Replacing $-\beta$ by β in (3.12) and adding the obtained equation to (3.12), we obtain

$$\sin 2\alpha\cos\beta\left((r_i - r_j)\sin^2\alpha + r_j - r_k\right)\left((\nabla_i\rho)_{ik} - (\nabla_j\rho)_{jk}\right) = 0.$$

The latter equality implies (3.7).

The formula (3.8) follows from (3.7) and (2.4). We derive (3.9) from (3.8) and (2.4).
Let $A = (a^1, a^2, a^3)$, $B = (b^1, b^2, b^3)$, $C = (c^1, c^2, c^3)$ be orthonormal vectors in $\mathbf{R^3}$.
We set

$$X = \sum_{i=1}^{3} a^i E_i, \quad Y = \sum_{i=1}^{3} b^i E_i, \quad Z = \sum_{i=1}^{3} c^i E_i, \quad U = E_4.$$

Taking into account (3.6), (3.7), (3.8) and (3.9), we obtain from (3.12) by straightforward computations that

$$\sum_{i=1}^{3}\left[a^i b^i c^i \left(\sum_{j=1}^{3}\left((b^j)^2 - (c^j)^2\right) r_j\right) + a^i\left((c^i)^2 - (b^i)^2\right)\left(\sum_{j=1}^{3} b^j c^j r_j\right)\right]. \tag{3.13}$$

$$[E_i(r_i) - E_i(r_4) - 2(\nabla_4\rho)_{4i}] = 0.$$

We replace $\bar{A} = (-a^1, a^2, a^3)$, $\bar{B} = (b^1, -b^2, -b^3)$, $\bar{C} = (c^1, -c^2, -c^3)$ by A, B, C in (3.13). Subtracting the obtained equation from (3.13), we get

$$a^1 \left[\left(\sum_{j=1}^{3} \left((b^j)^2 - (c^j)^2 \right) r_j \right) b^1 c^1 + \left(\sum_{j=1}^{3} b^j c^j r_j \right) \left((c^1)^2 - (b^1)^2 \right) \right] .$$

$$[E_1(r_1) - E_1(r_4) - 2(\nabla_4 \rho)_{41}] = 0.$$

It is easy to see that the latter equality is equivalent to the equation

$$(r_2 - r_3)\left(E_1(r_1) - E_1(r_4) - 2(\nabla_4 \rho)_{41}\right) = 0.$$

Similarly we get all other equalities in (3.10).
Conversely, let the identities (3.6), (3.7), (3.8), (3.9) and (3.10) hold. The condition (3.11) is equivalent to the equality

$$\sum_{s,p,r=1}^{4} a^s \left[b^p c^r \left(\sum_{j=1}^{4} \left((b^j)^2 - (c^j)^2 \right) r_j \right) + (c^p c^r - b^p b^r) \left(\sum_{j=1}^{4} b^j c^j r_j \right) \right.$$
$$\left. + c^p d^r \left(\sum_{j=1}^{4} b^j d^j r_j \right) - b^p d^r \left(\sum_{j=1}^{4} c^j d^j r_j \right) \right] (\nabla_s \rho)_{pr} = 0 \tag{3.14}$$

for any orthonormal vectors $A = (a^1, a^2, a^3, a^4)$, $B = (b^1, b^2, b^3, b^4)$, $C = (c^1, c^2, c^3, c^4)$, $D = (d^1, d^2, d^3, d^4)$ in $\mathbf{R^4}$. We check by direct computations that the formulas (3.6), (3.7), (3.8), (3.9) and (3.10) imply (3.14). **Q.E.D.**

Further, we have the following fife possibilities for the eigenvalues of the Ricci tensor on U_p:

A) $r_1 = r_2 = r_3 = r_4$,
B) $r_1 = r_2 = r_3 \neq r_4$,
C) $r_1 = r_2 \neq r_3 = r_4$,
D) $r_1 = r_2 \neq r_i, \quad i = 3, 4, \quad r_3 \neq r_4$,
E) $r_i \neq r_j, \quad i, j \in \{1, 2, 3, 4\},\ i \neq j$.

The case A. In this case (U_p, g) is a real space form.
The case B. We get from Proposition 3.1 that the following equalities hold

$$\omega_{ij}^4 = 0, \quad \text{for} \quad i, j \in \{1, 2, 3\}, i \neq j, \tag{3.15}$$

$$\omega_{11}^4 = \omega_{22}^4 = \omega_{33}^4, \tag{3.16}$$

$$\omega_{44}^1 = \omega_{44}^2 = \omega_{44}^3 = 0. \tag{3.17}$$

Since $(\mathbf{M}, g)$ is locally conformal flat, we have

$$0 = R(E_2, E_1, E_1, E_4) = E_2(\omega_{11}^4). \tag{3.18}$$

We obtain similarly

$$E_3(\omega_{11}^4) = E_1(\omega_{11}^4) = 0. \tag{3.19}$$

We set $\mathcal{F} = span\{E_1, E_2, E_3\}$, $\mathcal{F}^\perp = span\{E_4\}$, $\eta = \omega_{11}^4 E_4$. Let $\omega_{11}^4 = 0$ on U_p. Then $\mathcal{F}$ and $\mathcal{F}^\perp$ are autoparallel distributions. Hence, (U_p, g) is isometric to a Riemannian product

$\mathbf{B}\times\mathbf{N}$ of an 1-dimensional space B and a 3-dimensional Einstein space N. Hence, N is a real space form.

Let there exist $q\in U_p$ such that $\omega_{11}^4(q)\neq 0$. It follows from (3.15), (3.16), (3.17), (3.18) and (3.19) that we can apply a result of Hiepko (see [18], p.211) to conclude that the space (U_p,g) is locally isometric, around q, to a warped product of type $\mathbf{B}\times_{\mathbf{f}}\mathbf{N}$ with an 1-dimensional base $\mathbf{B^1}$ and a 3-dimensional leaf N^3 here $\mathbf{f}>0$ is a smooth function on $\mathbf{B^1}$. The Riemannian metric $g^{\mathbf{B^1}\times_{\mathbf{f}}\mathbf{N^3}}$ on $\mathbf{B^1}\times_{\mathbf{f}}\mathbf{N^3}$ is given by $g^{\mathbf{B^1}\times_{\mathbf{f}}\mathbf{N^3}}=g^{\mathbf{B^1}}+f^2g^{\mathbf{N^3}}$, where $g^{\mathbf{B^1}}$ and $g^{\mathbf{N^3}}$ are the Riemannian metrics on $\mathbf{B^1}$ and $\mathbf{N^3}$, respectively. Using the formulae for the Ricci eigenvalues of the warped product $\mathbf{B^1}\times_{\mathbf{f}}\mathbf{N^3}$ (see [23], p.204-210), we have

$$r_i=\frac{\tilde{\rho}_i}{\tilde{\mathrm{f}}^2}-2\left(\frac{E_4(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}\right)^2-\frac{E_4^2(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}},\quad i=1,2,3,\qquad r_4=-3\frac{E_4^2(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}},\tag{3.20}$$

where $\sim$ denotes the corresponding lift of a function on $\mathbf{B^1}$ or $\mathbf{N^3}$ to $\mathbf{B^1}\times_{\mathbf{f}}\mathbf{N^3}$ and ρ_i, $i=1,2,3$, are the Ricci eigenvalues of $\mathbf{N^3}$. The conditions $r_1=r_2=r_3$ imply that $\mathbf{N^3}$ is a 3-dimensional Einstein space and hence it is a real space form. Thus, III_1) follows.

The case C. In this case (3.9) and (3.10) imply that all Ricci eigenvalues are constant. Then we have locally symmetric Riemannian space since it is locally conformal flat [26],[20] and I) and II) follow.

The case D. The conditions (3.6), (3.7), (3.8), (3.9) and (3.10) of Proposition 3.1 are equivalent to the following equations

$$\omega_{ij}^k=0\quad\text{for}\quad\{j,k\}\neq\{1,2\},\tag{3.21}$$

$$\omega_{33}^k=\omega_{44}^k=0,\quad k=1,2,\tag{3.22}$$

$$E_k(r_1)=E_k(r_3)=E_k(r_4)=0,\quad k=1,2,\tag{3.23}$$

$$\omega_{11}^k=\omega_{22}^k,\quad k=3,4,\tag{3.24}$$

$$\begin{aligned}E_k(r_k)&=\frac{3}{2}E_k(r_1)=\frac{3}{2}E_k(r_j)=\\&=6(r_1-r_k)\omega_{11}^k=6(r_j-r_k)\omega_{jj}^k,\quad k,j\in\{3,4\},k\neq j.\end{aligned}\tag{3.25}$$

Since (U_p,g) is locally conformal flat, we have

$$0=R(E_2,E_1,E_1,E_3)=E_2(\omega_{11}^3).\tag{3.26}$$

We obtain analogously that

$$E_2(\omega_{11}^4)=0,\quad E_1(\omega_{11}^3)=E_1(\omega_{22}^3)=0,\quad E_1(\omega_{11}^4)=E_1(\omega_{22}^4)=0.\tag{3.27}$$

We set

$$\mathcal{F}=span\{E_1,E_2\},\quad \mathcal{F}^{\perp}=span\{E_3,E_4\},\quad \eta=\omega_{11}^3E_3+\omega_{11}^4E_4.$$

It follows from (3.21), (3.22), (3.24), (3.26), (3.27) and the mentioned above result of Hiepko ([18],p 211) that (U_p,g) is isometric to a warped product of type $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$ with a 2-dimensional base $\mathbf{B^2}$ and a 2-dimensional leaf N^2 where $\mathbf{f}>0$ is a smooth function on $\mathbf{B^2}$. The Riemannian metric $g^{\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}}$ on $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$ is given by $g^{\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}}=g^{\mathbf{B^2}}+f^2g^{\mathbf{N^2}}$, where $g^{\mathbf{B^2}}$ and $g^{\mathbf{N^2}}$ are the Riemannian metrics on $\mathbf{B^2}$ and $\mathbf{N^2}$, respectively. Let $\sim$ denote

a lift of a vector field or a function on $\mathbf{B^2}$ or $\mathbf{N^2}$ to $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$. Let $\mathcal{K}$ and $\mathcal{K}_{\mathrm{N}}$ be the sectional curvatures of $\mathbf{B^2}$ and $\mathbf{N^2}$, respectively and X_1, X_2 are orthonormal vector fields on $\mathbf{N^2}$. It follows by the well-known formulas of the warped product (see e.g. [23], p.204-210) that the vector fields

$$E_i = \frac{\tilde{X}_i}{\tilde{\mathrm{f}}}, \quad i = 1,2$$

are orthonormal eigenvectors of the Ricci tensor everywhere on $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$. The corresponding Ricci eigenfunctions are given by

$$r_1 = r_2 = \frac{\widetilde{\mathcal{K}_{\mathrm{N}}}}{\tilde{\mathrm{f}}^2} - \frac{\widetilde{\Delta \mathrm{f}}}{\tilde{\mathrm{f}}} - \frac{g(\widetilde{grad}(\mathrm{f}), \widetilde{grad}(\mathrm{f}))}{\tilde{\mathrm{f}}^2}, \tag{3.28}$$

where $\Delta\mathrm{f}$ is the Laplacian of $\mathbf{f}$.
There exist orthonormal smooth vector fields X_3, X_4 on $\mathbf{B^2}$ such that the vector fields $E_i = \tilde{X}_i, \quad i = 3,4$ are also orthonormal eigenvectors of the Ricci tensor everywhere on $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$. The corresponding Ricci eigenfunctions are determined by

$$r_3 = \tilde{\mathcal{K}} - 2\frac{H^{\mathrm{f}}(\widetilde{X_3}, \widetilde{X_3})}{\tilde{\mathrm{f}}}, \quad r_4 = \tilde{\mathcal{K}} - 2\frac{H^{\mathrm{f}}(\widetilde{X_4}, \widetilde{X_4})}{\tilde{\mathrm{f}}}. \tag{3.29}$$

We also have

$$H^{\mathrm{f}}(\widetilde{X_3}, \widetilde{X_4}) = 0, \tag{3.30}$$

where $H^{\mathrm{f}}(\widetilde{X_3}, \widetilde{X_4}) = \widetilde{X_3}\widetilde{X_3} - (\nabla_{\widetilde{X_3}}\widetilde{X_4})\mathrm{f}$ is the Hessian of f.

Lemma 3.2 *a) The warped product $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$ is locally conformal flat iff the following equality holds*

$$\frac{\widetilde{\mathcal{K}_{\mathrm{N}}}}{\tilde{\mathrm{f}}^2} + \frac{\widetilde{\Delta \mathrm{f}}}{\tilde{\mathrm{f}}} - \frac{g(\widetilde{grad}(\mathrm{f}), \widetilde{grad}(\mathrm{f}))}{\tilde{\mathrm{f}}^2} + \tilde{\mathcal{K}} = 0. \tag{3.31}$$

b) A locally conformal flat warped product $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$ with $r_1 = r_2 \neq r_i$, $i = 3,4$, $r_3 \neq r_4$ is a $\mathcal{P}$-space iff the condition (3.25) holds.

Proof of the Lemma 3.2. The condition a) follows from the warped product formulas (see e.g. [23], p.204-211) and (2.3) by direct computations. It is easy to verify that the equalities (3.21), (3.22) and (3.24) always hold on $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$. Formula (3.31) implies that if $\mathbf{B^2}\times_{\mathbf{f}}\mathbf{N^2}$ is a locally conformal flat then

$$\widetilde{\mathcal{K}_{\mathrm{N}}} = \mathcal{K}_{\mathrm{N}} = const \tag{3.32}$$

In this case (3.23) also holds. We apply Proposition 3.1 to complete the proof of Lemma 3.2. **Q.E.D.**

We shall prove that the conditions (3.31), (3.25) of Lemma 3.2 are equivalent to the following conditions

$$r_1 = 2\tilde{\mathcal{K}} + C, \quad r_3 = 2\tilde{\mathcal{K}} - D(x_3) + C, \quad r_4 = 3\tilde{\mathcal{K}} + D(x_3), \tag{3.33}$$

where C is a constant and $D(x_3)$ is a smooth function of x_3 such that $E_3(D) = -E_3(\tilde{\mathcal{K}})$,

$$\frac{\mathcal{K}_{\mathrm{N}}}{\tilde{\mathrm{f}}^2} - \left(\frac{E_3(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}\right)^2 - \left(\frac{E_4(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}\right)^2 = \frac{\tilde{\mathcal{K}} + C}{2}, \tag{3.34}$$

$$3E_3(\tilde{\mathcal{K}}) = E_3(r_3) = \frac{3}{2}E_3(r_4) = 6(r_1 - r_3)\omega_{11}^3 = 6(r_4 - r_3)\omega_{44}^3, \tag{3.35}$$

$$3E_4(\tilde{\mathcal{K}}) = E_4(r_4) = \frac{3}{2}E_4(r_3) = 6(r_1 - r_4)\omega_{11}^4 = 6(r_3 - r_4)\omega_{33}^4. \tag{3.36}$$

We have from (3.28), (3.29) and (3.31) that

$$r_3 + r_4 - r_1 = 3\tilde{\mathcal{K}}. \tag{3.37}$$

We obtain from (3.25) and (3.37) the equalities

$$\frac{2}{3}E_3(r_3) = E_3(r_4) = E_3(r_1) = 2E_3(\tilde{\mathcal{K}}), \tag{3.38}$$

$$\frac{2}{3}E_4(r_4) = E_4(r_3) = E_4(r_1) = 2E_4(\tilde{\mathcal{K}}). \tag{3.39}$$

Then (3.38) and (3.39) imply (3.33). We get using the formula $\Delta \mathrm{f} = H^{\mathrm{f}}(X_3, X_3) + H^{\mathrm{f}}(X_4, X_4)$ and (3.33) that

$$-\frac{\widetilde{\Delta \mathrm{f}}}{\tilde{\mathrm{f}}} = \frac{3\tilde{\mathcal{K}} + C}{2}.$$

We obtain (3.34) substituting the latter equality into (3.31). Conversely, (3.33), (3.34), (3.35) and (3.36) imply (3.31) and (3.25). The equalities (3.33), (3.38) and (3.39) imply that $r_3 = r_4$ iff D and $\tilde{\mathcal{K}}$ are constants.

It follows from the formulae (see [23])

$$\omega_{11}^3 = -\frac{E_3(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}, \quad \omega_{11}^4 = -\frac{E_4(\tilde{\mathrm{f}})}{\tilde{\mathrm{f}}}, \quad \tilde{\mathrm{f}} \neq const \tag{3.40}$$

that there exists a neighborhood V_q of almost every point $q \in U_p$ such that one of the following possibilities holds on V_q:

D1) $\omega_{11}^3 \neq 0$, $\omega_{11}^4 \neq 0$ everywhere on V_q,

D2) $\omega_{11}^3 \neq 0$, $\omega_{11}^4 = 0$ everywhere on V_q.

We shall describe below the metric on $\mathbf{B^2}$ and the function f on $\mathbf{B^2}$ in the both cases.

The case D1. We obtain $E_4(\tilde{\mathcal{K}}) \neq 0$ and $E_3(\tilde{\mathcal{K}}) \neq 0$ from (3.35), (3.36) and (3.40). We consider a local chart (V_q, x) of $\mathbf{B^2}$ with local coordinate fields $\frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_4}$. We set

$$a = g\left(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right), \quad b^2 = g\left(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3}\right), \qquad {}^{\mathrm{B}}\omega_{ii}^j = g(\nabla_{X_i} X_i, X_j), \quad i, j \in \{3, 4\}.$$

If h is a smooth function on $\mathbf{B^2}$ we denote by $h_{,i}\,, h_{,ij}\,, \quad i, j = 3, 4$ its first and second partial derivatives with respect to the local coordinates x_3, x_4. We have (changing the sign of X_3 if it is necessary) that

$$X_3 = \frac{\frac{\partial}{\partial x_3} - a\frac{\partial}{\partial x_4}}{\psi}, \quad X_4 = \frac{\partial}{\partial x_4}, \quad \text{where} \quad \psi = \sqrt{b^2 - a^2}.$$

Then, the metric g on $\mathbf{B^2}$ is given by

$$g = \psi^2 dx_3^2 + (a dx_3 + dx_4)^2. \tag{3.41}$$

We calculate that

$$ {}^{\mathrm{B}}\omega^3_{44} = \frac{a_{,4}}{\psi}, \quad {}^{\mathrm{B}}\omega^4_{33} = -\frac{\psi_{,4}}{\psi}. $$

$$ \mathcal{K} = X_3({}^{\mathrm{B}}\omega^3_{44}) + X_4({}^{\mathrm{B}}\omega^4_{33})^2 - ({}^{\mathrm{B}}\omega^4_{33})^2 - ({}^{\mathrm{B}}\omega^3_{44})^2. \tag{3.42} $$

Using the notations in the case $D1$, we get that the system of PDE (3.33), (3.34), (3.35) and (3.36) is equivalent to the following system of PDE

$$ \frac{\mathrm{f}_{,4}}{\mathrm{f}} = \frac{\mathcal{K}_{,4}}{2(\mathcal{K}+D-C)}, \tag{3.43} $$

$$ \frac{\psi_{,4}}{\psi} = \frac{\mathcal{K}_{,4}}{2(\mathcal{K}+2D-C)}, \tag{3.44} $$

$$ X_3(\mathcal{K}) = -X_3(D), \quad (i.e. \quad \mathcal{K}_{,3} - a\mathcal{K}_{,4} = -D_{,3}\,), \tag{3.45} $$

$$ \frac{X_3(\mathrm{f})}{\mathrm{f}} = \frac{D_{,3}}{2D\psi}, \tag{3.46} $$

$$ a_{,4} = -\frac{D_{,3}}{2(\mathcal{K}+2D-C)}, \tag{3.47} $$

$$ \frac{\mathcal{K}_{\mathrm{N}}}{\mathrm{f}^2} - \left(\frac{X_3(\mathrm{f})}{\mathrm{f}}\right)^2 - \left(\frac{\mathrm{f}_{,4}}{\mathrm{f}}\right)^2 = \frac{\mathcal{K}+C}{2}, \tag{3.48} $$

where $D(x_3) \neq 0, \quad D_{,3} \neq 0, \quad \mathcal{K} \neq C - 2D, \quad \mathcal{K} \neq C - D, \quad \mathcal{K}_{,4} \neq 0.$
The equations (3.42), (3.29) and (3.28) take the form

$$ \mathcal{K} = X_3\left(\frac{a_{,4}}{\psi}\right) + \left(-\frac{\psi_{,4}}{\psi}\right)_{,4} - \left(\frac{a_{,4}}{\psi}\right)^2 - \left(-\frac{\psi_{,4}}{\psi}\right)^2, \tag{3.49} $$

$$ \mathcal{K} - D + C = -2\left(\frac{X_3^2(\mathrm{f})}{\mathrm{f}} + \frac{\psi_{,4}}{\psi}.\frac{\mathrm{f}_{,4}}{\mathrm{f}}\right), \tag{3.50} $$

$$ 2\mathcal{K} + D = -2\left(\frac{\mathrm{f}_{,44}}{\mathrm{f}} - \frac{a_{,4}}{\psi}.\frac{X_3(\mathrm{f})}{\mathrm{f}}\right), \tag{3.51} $$

$$ 2\mathcal{K} + C = \frac{\mathcal{K}_{\mathrm{N}}}{\mathrm{f}^2} - \frac{\Delta \mathrm{f}}{\mathrm{f}} - \left(\frac{X_3(\mathrm{f})}{\mathrm{f}}\right)^2 - \left(\frac{\mathrm{f}_{,4}}{\mathrm{f}}\right)^2. \tag{3.52} $$

Moreover, the equation (3.30) has to be also satisfied.

We set $\mu = \mathcal{K} + D - C$. We get from (3.43) and (3.44) that

$$ \mathrm{f}^2 = H(x_3)\mu(x_3, x_4), \qquad \psi^2 = E(x_3)(\mu(x_3,x_4) + D(x_3)), \tag{3.53} $$

where $H(x_3)$ and $E(x_3)$ are smooth functions of x_3. Using (3.53), we obtain from (3.46) that

$$ H(x_3) = cD(x_3), \quad c = const. \tag{3.54} $$

We get using (3.53), (3.54) and (3.48) that (3.50) is equivalent to

$$ E = \frac{D_{,3}^2}{2D^2\left((D-C)^2 + \frac{2\mathcal{K}_{\mathrm{N}}}{cD} + e\right)}, \quad e = const. \tag{3.55} $$

The equation (3.49) follows from (3.45), (3.54) and (3.55). The equation (3.45) implies

$$a = \frac{\mu_{,3}}{\mu_{,4}}. \tag{3.56}$$

We obtain from (3.48) and (3.55) that

$$\mu_{,4}^2 = -\frac{2\mu}{\mu + D}\left(\left(\mu(\mu + C)^2 + e\right) - \frac{2\mathcal{K}_{\mathrm{N}}}{c}\right) \tag{3.57}$$

We replace (3.47) by $\left(\frac{\mu_{,3}}{\mu_{,4}}\right)_{,4} = -\frac{D_{,3}}{2(\mu+D)}$. The latter equality follows from (3.57). It is easy to verify that the equations (3.51),(3.52) and (3.30) are also consequences of (3.53)-(3.57). The equalities (3.53)-(3.57) together with (3.41) imply V) of Theorem 1.1. The spaces described in V) are locally conformal flat $\mathcal{P}$-space with $r_1 = r_2 \neq r_3 \neq r_4 \neq r_1$ by Lemma 3.2, (3.38) and (3.39). This completes the considerations in the case **D1**.

The case D2. We obtain $E_4(\tilde{\mathcal{K}}) = 0$ and $E_3(\tilde{\mathcal{K}}) \neq 0$ from (3.35), (3.36) and (3.40). Then (3.33), (3.38) and (3.39) imply

$$r_1 = 2\mathcal{K} + C, \quad r_3 = 3\mathcal{K} + A, \quad r_4 = 2\mathcal{K} - A + C,$$

where A is a constant. As in the previous case, we consider a chart (V_q, x) of $\mathbf{B^2}$ and keep the notations in **D1**. We proceed similarly as in the case **D1**. The system of PDE, corresponding to the system (3.43)-(3.52) in the case **D1**, can be written in the following way

$$\mathcal{K}_{,4} = 0, \qquad \mathrm{f}_{,4} = 0, \qquad \psi_{,4} = 0,$$

$$\frac{X_3(\mathrm{f})}{\mathrm{f}} = \frac{\mathcal{K}_{,3}}{2\psi(\mathcal{K} + A - C)}, \tag{3.58}$$

$$a_{,4} = -\frac{\mathcal{K}_{,3}}{2(\mathcal{K} + 2A - C)}, \tag{3.59}$$

$$\frac{\mathcal{K}_{\mathrm{N}}}{\mathrm{f}^2} - \left(\frac{X_3(\mathrm{f})}{\mathrm{f}}\right)^2 = \frac{\mathcal{K} + C}{2}, \tag{3.60}$$

$$\mathcal{K} = X_3\left(\frac{a_{,4}}{\psi}\right) - \left(\frac{a_{,4}}{\psi}\right)^2, \tag{3.61}$$

$$\mathcal{K} - A + C = \frac{2a_{,4}}{\psi}.\frac{X_3(\mathrm{f})}{\mathrm{f}}, \tag{3.62}$$

$$2\mathcal{K} + A = -2\frac{X_3^2(\mathrm{f})}{\mathrm{f}}, \tag{3.63}$$

$$2\mathcal{K} + C = \frac{\mathcal{K}_{\mathrm{N}}}{\mathrm{f}^2} - \frac{\Delta \mathrm{f}}{\mathrm{f}} - \left(\frac{X_3(\mathrm{f})}{\mathrm{f}}\right)^2, \tag{3.64}$$

where $\mathcal{K}_{,3} \neq 0, \quad \mathcal{K} \neq C - 2A, \quad \mathcal{K} \neq C - A, \quad A \neq 0.$

We get from (3.58), (3.59) and (3.62) that

$$\psi^2 = -\frac{\mathcal{K}_{,3}^2}{2(\mathcal{K} + 2A - C)(\mathcal{K}^2 - (A - C)^2)}. \tag{3.65}$$

The equation (3.61) is a consequence of (3.59) and (3.65). The equation (3.58) implies

$$\mathrm{f}^2 = c(K + A - C), \quad c = const. \neq 0. \tag{3.66}$$

We obtain from (3.60) and (3.66) that

$$C = A - \frac{K_N}{cA}. \tag{3.67}$$

Using (3.67), we obtain from (3.59) that

$$a = -\frac{\mathcal{K}_{,3}}{2\left(\mathcal{K} + \frac{\mathcal{K}_{\mathrm{N}}}{cA} + A\right)} x_4 + \alpha(x_3), \tag{3.68}$$

where $\alpha(x_3)$ is a smooth function of x_3. The equations (3.63),(3.64) and (3.30) are also consequences of (3.65)-(3.68).

We substitute (3.67) into (3.65) and (3.66). Substituting the obtained equations and (3.68) into (3.41), we get the spaces described in IV) of Theorem 1.1. These spaces are locally conformal flat $\mathcal{P}$-space with $r_1 = r_2 \neq r_3 \neq r_4 \neq r_1$ by Lemma 3.2, (3.38) and (3.39). This completes the considerations in the case **D2**.

The case E. In this case, the formulae (3.6), (3.7), (3.8), (3.9) and (3.10) of Proposition 3.1 are equivalent to

$$\omega_{ij}^k = 0, \tag{3.69}$$

$$(r_j - r_k)\omega_{jj}^k = (r_i - r_k)\omega_{ii}^k, \tag{3.70}$$

$$E_k(r_i) = E_k(r_j), \tag{3.71}$$

$$E_k(r_k) = \frac{3}{2}E_k(r_i), \tag{3.72}$$

$$E_k(r_i) = 4(r_i - r_k)\omega_{ii}^k. \tag{3.73}$$

Let $\mathcal{F}_i$ be the orthonormal complement to E_i, $i = 1, 2, 3, 4$. The equality (3.69) implies that the distributions $\mathcal{F}_i$ are integrable. Then (decreasing, if it is necessary, the neighborhood U_p of the point p) we can find a chart (U_p, x) of $(\mathbf{M}, g)$, (see [15, 13]) such that the metric g is given by

$$g = \mu_1^2 dx_1^2 + \mu_2^2 dx_2^2 + \mu_3^2 dx_3^2 + \mu_4^2 dx_4^2, \tag{3.74}$$

where the functions $\mu_i = \mu_i(x_1, x_2, x_3, x_4)$ are smooth and strictly positive and we have

$$X_i := \frac{\partial}{\partial x_i} = \mu_i E_i, \quad i = 1, 2, 3, 4. \tag{3.75}$$

We set

$$\nu_s = \ln(\mu_s), \quad \nu_{s,t} = \frac{\partial}{\partial x_t}\nu_s = X_t(\nu_s), \quad \nu_{s,tr} = \frac{\partial^2}{\partial x_t \partial x_r}\nu_s,$$

$$\nabla_{X_s} X_t = \sum_{r=1}^{4} \Gamma_{st}^r X_r, \qquad R(X_t, X_r, X_s) = \sum_{u=1}^{4} R_{str}^u X_u$$

We calculate using (3.74) that

$$\Gamma^k_{ij} = 0, \quad \Gamma^k_{ik} = \Gamma^k_{ki} = \nu_{k,i}, \quad \Gamma^k_{ii} = -\frac{\mu_i^2}{\mu_k^2}\nu_{i,k}, \quad \Gamma^i_{ii} = \nu_{i,i}, \tag{3.76}$$

$$\begin{aligned} R^i_{jij} = \mu_j^2 \Bigg[& \frac{1}{\mu_j^2}(\nu_{i,j}\nu_{j,j} - \nu_{i,j}^2 - \nu_{i,jj}) + \frac{1}{\mu_i^2}(\nu_{j,i}\nu_{i,i} - \nu_{j,i}^2 - \nu_{j,ii}) \\ & - \frac{1}{\mu_k^2}\nu_{i,k}\nu_{j,k} - \frac{1}{\mu_l^2}\nu_{i,l}\nu_{j,l} \Bigg], \end{aligned} \tag{3.77}$$

$$R^i_{kij} = -\nu_{i,jk} + \nu_{i,j}\nu_{j,k} + \nu_{i,k}\nu_{k,j} - \nu_{i,j}\nu_{i,k}. \tag{3.78}$$

Lemma 3.3 *If the formulas (3.69)-(3.73) are valid then the functions $\nu_1, \nu_2, \nu_3, \nu_4$ satisfy the following system of non-linear PDE on U_p.*

$$\nu_{i,j}\nu_{j,k} + \nu_{i,k}\nu_{k,j} - \nu_{i,j}\nu_{i,k} = 0, \tag{3.79}$$

$$\nu_{i,jk} = 0, \tag{3.80}$$

$$\nu_{i,ij} + 2\nu_{i,j}\nu_{j,i} = 0. \tag{3.81}$$

Proof of the Lemma: It follows from (3.75) and (3.76) that

$$\nu_{i,k} + \mu_k \omega^k_{ii} = 0 \tag{3.82}$$

Then, the equality (3.70) is equivalent to

$$(r_i - r_k)\nu_{i,k} = (r_j - r_k)\nu_{j,k}. \tag{3.83}$$

If $\nu_{i,k} = 0$ on U_p, we have from (3.82) that $\nu_{j,k} = 0$ which implies (3.79).
Let there exist a point $q \in U_p$ such that $\nu_{i,k}(q) \neq 0$. The formula (3.79) follows from (3.83) and the equality

$$\frac{r_i - r_k}{r_j - r_k} - 1 = -\frac{r_i - r_j}{r_k - r_j}.$$

We obtain $R^i_{kij} = 0$ from (2.3) and (3.75). The latter equality, together with (3.78) and (3.79), implies (3.80).
If $\nu_{i,j} = 0$ on U_p then (3.81) is valid. Let $\nu_{i,j} \neq 0$. It follows from (3.83) that $\nu_{k,j} \neq 0$ and the following equalities hold

$$\frac{r_k - r_j}{r_i - r_j} = \frac{\nu_{i,j}}{\nu_{k,j}}. \tag{3.84}$$

We have from (3.72), (3.73), (3.75) and (3.82) that

$$X_i(r_j - r_i) = -\frac{1}{2}X_i(r_j) + 2(r_j - r_i)\nu_{j,i}. \tag{3.85}$$

Using (3.71), (3.84) and (3.85), we obtain consequently

$$X_i(r_j - r_i) = X_i(r_k - r_i) = (r_j - r_i)X_i\left(\frac{r_k - r_i}{r_j - r_i}\right) + \frac{r_k - r_i}{r_j - r_i}X_i(r_j - r_i),$$

$$\left(1-\frac{r_k-r_i}{r_j-r_i}\right)X_i(r_j-r_i)=(r_j-r_i)X_i\left(\frac{r_k-r_i}{r_j-r_i}\right),$$

$$2\left(1-\frac{r_k-r_i}{r_j-r_i}\right)(r_j-r_i)\nu_{j,i}=-(r_j-r_i)X_i\left(1-\frac{r_k-r_i}{r_j-r_i}\right),$$

$$2\frac{r_j-r_k}{r_j-r_i}=-X_i\left(\frac{r_j-r_k}{r_j-r_i}\right),$$

$$2\frac{\nu_{i,j}}{\nu_{k,j}}\nu_{j,i}=-X_i\left(\frac{\nu_{i,j}}{\nu_{k,j}}\right)$$

The latter equality together with (3.80) implies (3.81). So the whole Lemma is proved. **Q.E.D.**

The system (3.79)-(3.81) of non-linear PDE is well known as a system of Stäckel: it arises from a quantum mechanical problems; on the other hand it is shown in [4] that this system describes 3-dimensional $\mathcal{P}$-spaces with distinct Ricci eigenvalues; it is shown in [21] that the local description of Riemannian 3-manifolds with constant eigenvalues of the natural skew-symmetric curvature operator on each unit circle also reduces to this system. The Stäckel system is solved in Euclidean 3-space by Weinacht in [27], W.Blaschke in [8]. L.P.Eisenhart solved in [13] this system completely in dimension three and obtained solutions in higher dimensions. In [14] L.P.Eisenhart classified all three dimensional Stäckel systems in locally conformal flat 3-space. In order to complete the proof of Theorem 1.1 we have to solve the Stäckel system in dimension 4. Following the ideas of [13] we obtain a complete solution of the Stäckel system in dimension 4, namely we have

Lemma 3.4 *The Stäckel system (3.79)-(3.81) has the following* 10 *kinds of solutions in dimension* 4*:*

$\mathbf{I_1}$. $\mu_1^2=1,\mu_2^2=\eta(x_1),\mu_3^2=\eta(x_1)\psi(x_2),\mu_4^2=\eta(x_1)\phi(x_2)$*, where* $\eta'\neq 0$*,* η *is a function of* x_1*,* ψ *and* ϕ *are functions of* x_2*.*

$\mathbf{I_2}$. $\mu_1^2=1,\mu_2^2=\phi(x_1)(\xi(x_2)-\zeta(x_3))(\xi(x_2)-\eta(x_4)),\mu_3^2=\phi(x_1)(\zeta(x_3)-\xi(x_2))(\zeta(x_3)-\eta(x_4)),\mu_4^2=\phi(x_1)(\eta(x_4)-\xi(x_2))(\eta(x_4)-\zeta(x_3))$*, where* ψ,ϕ,η,ζ *are functions of one variable and* $\zeta'\neq 0,\eta'\neq 0$*.*

$\mathbf{I_3}$. $\mu_1^2=\eta(x_1),\mu_2^2=x_1\xi(x_2),\mu_3^2=\mu_4^2=x_1x_2(\phi(x_3)+\psi(x_4))$*, where* η,ξ,ψ,ϕ *are functions of one variable.*

$\mathbf{I_4}$. $\mu_1^2=\mu_2^2=\phi(x_1)+\psi(x_2),\mu_3^2=\mu_4^2=\xi(x_3)+\eta(x_4)$*, where* ϕ,ψ,ξ,η *are functions of one variable.*

$\mathbf{I_5}$. $\mu_1^2=\mu_2^2=(\xi(x_1)-\eta(x_2)),\mu_3^2=\mu_4^2=\xi(x_1)\eta(x_2)(\psi(x_3)+\phi(x_4))$*, where* ϕ,ψ,ξ,η *are functions of one variable and* $\xi'\neq 0,\eta'\neq 0$*.*

$\mathbf{I_6}$. $\mu_1^2=\phi(x_1),\mu_2^2=\psi(x_1),\mu_3^2=\mu_4^2=x_1(\xi(x_3)+\eta(x_4))$*, where* ϕ,ψ,ξ,η *are functions of one variable and* $\phi>0,\psi>0$

$\mathbf{I_7}$. $\mu_1^2=1,\mu_2^2=\phi(x_1),\mu_3^2=\psi(x_1),\mu_4^2=\eta(x_1)$*, where* η,ψ,ϕ *are functions of* x_1*.*

$\mathbf{I_8}$. $\mu_i^2=\phi_i(x_i)|x_i-x_j||x_i-x_k||x_i-v_l|$ *for* $i,j,k,l\in\{1,2,3,4\}$ *distinct, where* ϕ_i *is a function of* x_i*.*

$\mathbf{I_9}$. $\mu_1^2=x_2x_3x_4,\mu_i^2=\phi_i(x_i)|x_i-x_j||x_i-x_k|$ *for* $i,j,k\in\{2,3,4\}$ *distinct, where* ϕ_i *is a function of* x_i*.*

$\mathbf{I_{10}}$. $\mu_1^2=|x_3-b||x_4-b|,\mu_2^2=x_3x_4,\mu_i^2=\phi_i(x_i)|x_i-x_j|$ *for* $i,j\in\{3,4\}$ *distinct, where* ϕ_i *is a function of* x_i*.*

Proof of the Lemma. We follow the considerations of L.P.Eisenhart. In [13] he showed how one could obtain all solutions of the Stäckel system in an arbitrary dimension and he found four kinds of explicit solutions in dimension three. We apply his considerations in [13],p.291 − 294 to the four dimensional case. The formula (3.14) of [13] implies the case $\mathbf{I_8}$. All solutions are given by formula (3.9) of [13]. We write down explicitly the conditions (3.10) and (3.11) of [13] for four distinct indices. All possible cases are given by the formulas (3.17) − (3.19) of [13]. We examine carefully each of these cases and we get by a lengthy but straitforward computations that all the remaining solutions of the Stäckel system in dimension four can be described by $\mathbf{I_1} - \mathbf{I_7}, \mathbf{I_9}, \mathbf{I_{10}}$, using a suitable changes of the local coordinates. **Q.E.D.**

To complete the proof of Theorem 1.1 we have to pick out those of the spaces $\mathbf{I_1} - \mathbf{I_{10}}$ which are locally conformal flat $\mathcal{P}$-spaces and have four distinct eigenvalues of the Ricci operator. We have

Lemma 3.5 *Let* $(\mathbf{M}, \mathbf{g})$ *be a* 4*-dimensional Riemannian manifold with the metric of the form (3.74) for which the conditions of Lemma 3.3 hold. Then*

i) $(\mathbf{M}, g)$ *is locally conformal flat iff*

$$R(E_i, E_l, E_l, E_i) + R(E_k, E_j, E_j, E_k) = R(E_i, E_k, E_k, E_i) + R(E_j, E_l, E_l, E_j). \tag{3.86}$$

ii) If $(\mathbf{M}, g)$ *is locally conformal flat with four distinct Ricci eigenvalues then it is a* $\mathcal{P}$*-space iff*

$$\frac{2}{3} X_i(r_i) = X_j(r_j) = 4(r_i - r_j)\nu_{j,i}. \tag{3.87}$$

Proof of the Lemma. The proof of i) follows from (2.3) and (3.74) by direct computations. The proof of ii) is a consequence of (3.69)-(3.73) and the conditions of the Lemma. **Q.E.D.**

Spaces of type $\mathbf{I_1}, \mathbf{I_2}$ and $\mathbf{I_3}$. A space of each of these three types is a warped product of a 1-dimensional base B and a 3-dimensional leaf N. Since it is locally conformal flat then the leaf N is a space of constant sectional curvature. Then (3.20) implies that the number of distinct Ricci eigenvalues is at most two.

Spaces of type $\mathbf{I_4}, \mathbf{I_5}, \mathbf{I_6}$ and $\mathbf{I_{10}}$ **with** $b = 0$. A space of any of these four types can be regarded as a warped (or Riemannian) product of two Riemannian surfaces. It follows by the formulas for the Ricci tensor of a warped product (3.28) that the number of distinct Ricci eigenvalues is at most three.

Spaces of type $\mathbf{I_7}$. Using (3.75) and (3.77), we calculate consequently

$$R(E_1, E_j, E_j, E_1) = -\nu_{j,1}^2 - \nu_{j,11}, \quad R(E_i, E_j, E_j, E_i) = -\nu_{i,1}\nu_{j,1} \tag{3.88}$$

for $i, j \in \{2, 3, 4\}$.

$$r_1 = -\nu_{2,1}^2 - \nu_{3,1}^2 - \nu_{4,1}^2 - \nu_{2,11} - \nu_{3,11} - \nu_{4,11}, \tag{3.89}$$

$$r_i = -\nu_{i,11} - \nu_{i,1}(\nu_{2,1} + \nu_{3,1} + \nu_{4,1}), \quad i = 2, 3, 4, \tag{3.90}$$

Because of (3.86), we get from (3.90) that

$$r_i - r_j = 2\nu_{k,1}(\nu_{j,1} - \nu_{i,1}), \quad i, j, k \in \{2, 3, 4\}. \tag{3.91}$$

It follows from (3.71) that $X_1(r_i - r_j) = 0$. The latter equality and (3.91) imply

$$(\nu_{j,11} - \nu_{i,11})\nu_{k,1} + (\nu_{j,1} - \nu_{i,1})\nu_{k,11} = 0. \tag{3.92}$$

We obtain using (3.88) that the equality (3.86) for $l = 1$ is equivalent to

$$\nu_{j,11} - \nu_{i,11} = (\nu_{j,1} - \nu_{i,1})(\nu_{k,1} - \nu_{j,1} - \nu_{i,1}) \tag{3.93}$$

We get from (3.91), (3.92) and (3.93) that

$$\nu_{i,1} \neq 0, \quad \nu_{i,1} \neq \nu_{j,1}, \tag{3.94}$$

$$(\nu_{i,1} + \nu_{j,1} - \nu_{k,1})\nu_{k,1} - \nu_{k,11} = 0, \tag{3.95}$$

where i, j, k belong to the set $\{2, 3, 4\}$.

To complete the considerations in the case $\mathbf{I_7}$ we have to solve the system of PDE (3.94) and (3.95). By direct computations it follows that the conditions (3.95) are equivalent to the following equations

$$(\nu_{2,1}\nu_{3,1})_{,1} = (\nu_{2,1}\nu_{4,1})_{,1} = (\nu_{3,1}\nu_{4,1})_{,1} = 2\nu_{2,1}\nu_{3,1}\nu_{4,1}. \tag{3.96}$$

The latter equalities are equivalent to

$$\nu_{3,1}(\nu_{2,1} - \nu_{4,1}) = D, \quad \nu_{4,1}(\nu_{4,1} - \nu_{3,1}) = F, \quad \nu_{3,1}\nu_{4,1} = Pe^{2\nu_2} \tag{3.97}$$

where D,F, P are constants. We obtain from the symmetries of (3.95) that

$$\nu_{2,1}\nu_{3,1} = Qe^{2\nu_4}, \quad \nu_{2,1}\nu_{4,1} = Re^{2\nu_3}, \tag{3.98}$$

for some constants Q, R. Then we have

$$\begin{gathered} Qe^{2\nu_4} - Pe^{2\nu_2} = D, \quad Re^{2\nu_3} - Pe^{2\nu_2} = F, \\ \nu_3 = \frac{1}{2}\ln\left(\frac{Pe^{2\nu_2} + F}{R}\right), \quad \nu_4 = \frac{1}{2}\ln\left(\frac{Pe^{2\nu_2} + D}{Q}\right), \\ \nu_{3,1} = \frac{Pe^{2\nu_2}\nu_{2,1}}{Pe^{2\nu_2} + F}, \quad \nu_{4,1} = \frac{Pe^{2\nu_2}\nu_{2,1}}{Pe^{2\nu_2} + D}. \end{gathered} \tag{3.99}$$

Substituting the latter two equalities in the third equation of (3.97), we obtain

$$\nu_{2,1}^2 = \frac{(Pe^{2\nu_2} + F)(Pe^{2\nu_2} + D)}{Pe^{2\nu_2}}. \tag{3.100}$$

It follows from (3.94), (3.97) and (3.98) that

$$F \neq D, \quad FDPQR \neq 0. \tag{3.101}$$

Conversely, the formulas (3.99), (3.100) and (3.101) imply (3.97), (3.98) and (3.94).
If we set $a = \frac{F}{P}, \quad b = \frac{D}{P}, \quad r = \frac{P}{R}, \quad q = \frac{P}{Q}$, than we obtain that (3.99), (3.100) and (3.101) are equivalent to $\mu_3^2 = r(\mu_2^2 + a), \quad \mu_4^2 = q(\mu_2^2 + b), \quad \mu_{2,1}^2 = (\mu_2^2 + a)(\mu_2^2 + b), \quad r \neq$

0, $\quad q \neq 0, \quad a \neq 0, \quad b \neq 0, \quad a \neq b$. Setting $\phi = \mu_2$ we get the metrics described in the case VI).

Conversely, for any metrics described in VI), we get from (3.88) that (3.86) is satisfied. We calculate for the Ricci eigenvalues using (3.89) and (3.90) that

$$r_1 = -6\phi^2 - 2a - 2b, \quad r_2 = -4\phi^2 - 2a - 2b,$$
$$r_3 = -4\phi^2 - 2b, \quad r_4 = -4\phi^2 - 2a. \tag{3.102}$$

The equalities (3.102) imply that all the Ricci eigenvalues are distinct and (3.87) is satisfied. Then Lemma 3.5 proves VI).

Spaces of type $\mathbf{I_8}$. We set $\psi_s = \frac{1}{\phi_s(x_s)}, s = 1,2,3,4$. We multiply (3.86) by $\frac{1}{(x_i - x_l)(x_j - x_k)}$. We get from the obtained equation and (3.77) the equation

$$\begin{aligned}
&\frac{1}{(x_j - x_i)^2(x_j - x_k)^2(x_j - x_l)^2}\left(-\psi_j' + 2\psi_j\left(\frac{1}{x_j - x_i} + \frac{1}{x_j - x_l} + \frac{1}{x_j - x_k}\right)\right) + \\
&\frac{1}{(x_i - x_j)^2(x_i - x_k)^2(x_i - x_l)^2}\left(-\psi_i' + 2\psi_i\left(\frac{1}{x_i - x_j} + \frac{1}{x_i - x_l} + \frac{1}{x_i - x_k}\right)\right) + \\
&\frac{1}{(x_k - x_i)^2(x_k - x_j)^2(x_k - x_l)^2}\left(-\psi_k' + 2\psi_k\left(\frac{1}{x_k - x_i} + \frac{1}{x_k - x_l} + \frac{1}{x_k - x_j}\right)\right) + \\
&\frac{1}{(x_l - x_i)^2(x_l - x_k)^2(x_l - x_j)^2}\left(-\psi_l' + 2\psi_l\left(\frac{1}{x_l - x_i} + \frac{1}{x_l - x_j} + \frac{1}{x_l - x_k}\right)\right) = 0
\end{aligned} \tag{3.103}$$

We multiply (3.103) by $(x_l - x_j)^2$. Differentiating the obtained equation with respect to x_j we multiply the result by $(x_l - x_j)^2$. We differentiate the obtained result with respect to x_j. Multiplying the obtained equation by $(x_l - x_j)^3(x_l - x_k)^3$, we get a polynomial of degree eight in x_l. Each of its coefficients must vanish. Equating to zero the coefficient of the highest degree, we obtain

$$\begin{aligned}
&\left[\frac{1}{(x_j - x_i)^2(x_j - x_k)^2}\left(-\psi_j' + 2\psi_j\left(\frac{1}{x_j - x_i} + \frac{1}{x_j - x_k}\right)\right)\right]'' + \\
&\frac{1}{(x_i - x_k)^2}\left[\frac{1}{(x_i - x_j)^2}\left(-\psi_i' + 2\psi_i\left(\frac{1}{x_i - x_j} + \frac{1}{x_i - x_k}\right)\right)\right]'' + \\
&\frac{1}{(x_k - x_i)^2}\left[\frac{1}{(x_k - x_j)^2}\left(-\psi_k' + 2\psi_k\left(\frac{1}{x_k - x_i} + \frac{1}{x_k - x_j}\right)\right)\right]'' = 0
\end{aligned}$$

We multiply the latter equation by $(x_j - x_k)^4$. Differentiating the obtained equation with respect to x_j, we multiply the result by $(x_j - x_k)^2$ and differentiating the obtained equation with respect to x_j we obtain a polynomial in x_k. Equating to zero the coefficient of the highest degree, we get

$$\left[\frac{1}{(x_j - x_i)^2}\left(-\psi_j' + \frac{2\psi_j}{x_j - x_i}\right)\right]^{IV} + \left[\frac{1}{(x_i - x_j)^2}\left(-\psi_i' + \frac{2\psi_i}{x_i - x_j}\right)\right]^{IV} = 0$$

We multiply this equation by $(x_i - x_j)^6$, differentiating the obtained equation with respect to x_j, we multiply the result by $(x_i - x_j)^7$. Differentiating the obtained equation with respect to x_j, we get a polynomial of x_i. Equating to zero the coefficient of the highest degree we derive $\psi_j^{VII} = 0$. Hence, the functions $\psi_s, s = 1, 2, 3, 4$ are polynomials of degree six. Multiplying (3.103) by $(x_j - x_i)^3$ and taking $x_j = x_i \neq x_k \neq x_l \neq x_i$, we get from the obtained equation that $\psi_j = \psi_i$. So, we may suppose

$$\psi_s(x) = a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 + a_0, \quad s = 1, 2, 3, 4. \tag{3.104}$$

We calculate using (3.104) and (3.77) that

$$R(E_i, E_j, E_j, E_i) = -\left[2(x_j - x_i) + x_k + x_l\right] \frac{a_6}{4} - \frac{a_5}{4}, \tag{3.105}$$

$$r_i = -a_6\left(\frac{3}{2} x_i + x_j + x_k + x_l\right) - \frac{3}{4} a_5, \quad r_i - r_j = -\frac{a_6}{2}(x_i - x_j).$$

Using (3.105), it is easy to verify that the conditions of Lemma 3.5 are satisfied, i.e we have a locally conformal flat P-space with four distinct Ricci eigenvalues iff $a_6 \neq 0$. So, we prove VII).

Spaces of type $\mathbf{I_9}$. We set $\psi_s = \frac{1}{\phi_s(x_s)}, s = 2, 3, 4$. Throughout the considerations in this case the indices j, k, l will stand for any distinct indices from the set $\{2, 3, 4\}$. We get from (3.86) and (3.77) the equation

$$\begin{aligned}
&\frac{1}{x_j (x_j - x_k)^2 (x_j - x_l)^2} \left(-\psi_j' + \psi_j \left(\frac{1}{x_j} + 2\frac{1}{x_j - x_l} + 2\frac{1}{x_j - x_k} \right) \right) + \\
&\frac{1}{x_k (x_k - x_j)^2 (x_k - x_l)^2} \left(-\psi_k' + 2\psi_k \left(\frac{1}{x_k} + 2\frac{1}{x_k - x_l} + 2\frac{1}{x_k - x_j} \right) \right) + \\
&\frac{1}{x_l (x_l - x_k)^2 (x_l - x_j)^2} \left(-\psi_l' + 2\psi_l \left(\frac{1}{x_l} + 2\frac{1}{x_l - x_j} + 2\frac{1}{x_l - x_k} \right) \right) = 0.
\end{aligned} \tag{3.106}$$

We multiply (3.106) by $\frac{(x_j - x_l)^2 (x_l - x_k)^2}{(x_j - x_k)}$. Differentiating the obtained equation with respect to x_j we multiply the result by $(x_l - x_j)^2$. We differentiate the obtained result with respect to x_j and multiplying the obtained equation by $(x_k - x_l)$, we get a polynomial of degree six in x_l. Each of its coefficients must vanish. Equating to zero the coefficient of the highest degree, we obtain

$$\begin{aligned}
&\left[\frac{1}{x_j (x_j - x_k)^2} \left(-\psi_j' + \psi_j \left(\frac{1}{x_j} + 2\frac{1}{x_j - x_k} \right) \right) \right]'' + \\
&\left[\frac{1}{x_k (x_k - x_j)^2} \left(-\psi_k' + \psi_k \left(\frac{1}{x_k} + 2\frac{1}{x_k - x_j} \right) \right) \right]'' = 0
\end{aligned}$$

Differentiating the above equation with respect to x_j, we multiply the result by $(x_j - x_k)^2$. We differentiate the obtained equation with respect to x_j. Multiplying the obtained equation by x_k, we get a polynomial of x_k of degree five. Equating to zero the coefficient of the highest degree we derive $\left(\frac{\psi_j}{x_j}\right)^V = 0$. Hence, the functions $\psi_s, s = 2, 3, 4$ are

polynomials of degree five having a root equal to zero. Multiplying (3.106) by $(x_j - x_k)^2$ and taking $x_j = x_k \neq x_l$, we get from the obtained equation that $\psi_j = \psi_k$. So, we may suppose

$$\psi_s(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x, \quad s = 2, 3, 4. \tag{3.107}$$

We calculate using (3.107) and (3.77) that

$$\begin{aligned}
R(E_j, E_k, E_k, E_j) &= -\frac{1}{4}\left(a_5\left(2(x_j + x_k) + x_l\right) + a_4\right), \\
R(E_1, E_j, E_j, E_1) &= -\frac{1}{4}\left(a_5\left(2x_j + x_k + x_l\right) + a_4\right), \\
r_1 = -a_5(x_j + x_k + x_l) - \frac{3a_4}{4}, &\qquad r_j = -a_5\left(\frac{3}{2}x_j + x_k + x_l\right) - \frac{3a_4}{4}.
\end{aligned} \tag{3.108}$$

Using (3.108), it is easy to verify that the conditions of Lemma 3.5 are satisfied, i.e we have a locally conformal flat P-space with four distinct Ricci eigenvalues iff $a_5 \neq 0$. Thus, we have proved VIII).

Spaces of type $\mathbf{I_{10}}$ with $\mathbf{b \neq 0}$. We set $\psi_s = \frac{1}{\phi_s(x_s)}$, $s = 3, 4$. Throughout the considerations in this case the indices k, l will stand for any distinct indices from the set $\{3, 4\}$. We get from (3.86) and (3.77) the equation

$$\begin{aligned}
&\frac{1}{x_k(x_k - b)}\left(-\psi_k' + \psi_k\left(\frac{1}{x_k} + 2\frac{1}{x_k - x_l} + \frac{1}{x_k - b}\right)\right) + \\
&\frac{1}{x_l(x_l - b)}\left(-\psi_l' + \psi_l\left(\frac{1}{x_l} + 2\frac{1}{x_l - x_k} + \frac{1}{x_l - b}\right)\right) = 0.
\end{aligned} \tag{3.109}$$

We differentiate (3.109) with respect to x_k. We multiply the obtained result by $(x_k - x_l)^2$. Differentiating the obtained equation by x_k, we get $\left[\frac{\psi_k}{x_k(x_k - b)}\right]^{III} = 0$. Multiplying (3.109) by $(x_k - x_l)^2$ and taking $x_k = x_l$, we derive from the obtained equation that $\psi_3 = \psi_4$. So, we may suppose

$$\psi_s(x) = (x - b)(a_3x^3 + a_2x^2 + a_1x), \quad s = 3, 4. \tag{3.110}$$

We calculate using (3.110) and (3.77) that

$$\begin{aligned}
R(E_1, E_2, E_2, E_1) &= -\frac{1}{4}\left(a_3\left(x_3 + x_4\right) + a_2\right), \\
R(E_1, E_k, E_k, E_1) &= -\frac{1}{4}\left(a_3\left(2x_k + x_l\right) + a_2\right), \\
R(E_3, E_4, E_4, E_3) &= -\frac{1}{4}\left(a_3 2\left(x_3 + x_4\right) + a_2 - ba_3\right), \\
R(E_2, E_k, E_k, E_2) = -\frac{1}{4}\left(a_3\left(2x_k + x_l\right) + a_2 - ba_3\right), &\quad r_1 = -a_3(x_k + x_k + x_l) - \frac{3a_2}{4}, \\
r_2 = -a_3(x_k + x_k + x_l) - \frac{3a_2}{4} + \frac{ba_3}{2}, &\qquad r_k = -a_3\left(\frac{3}{2}x_k + x_j\right) - \frac{3a_2}{4} + \frac{ba_3}{2}.
\end{aligned} \tag{3.111}$$

Using (3.111), it is easy to verify that the conditions of Lemma 3.5 are satisfied, i.e we have a locally conformal flat P-space with four distinct Ricci eigenvalues iff $a_3 \neq 0$. Thua, we get IX which completes the proof of Theorem 1.1 **Q.E.D.**

4 Proof of Theorem 1.2

We begin with

Lemma 4.1 *Every locally conformal flat* 4*-dimensional* $\mathcal{Q}$*-space is a* $\mathcal{P}$*-space.*

Proof. Let (**M**,g) be an $\mathcal{Q}$-space of dimension 4. Evaluating explicitly the right hand term of (1.1) in dimension 4 we get that the equality (1.1) is equivalent to

$$(\nabla_X\rho)(Y,Z) = \frac{2}{9}X(s)g(Y,Z) + \frac{1}{18}Y(s)g(X,Z) + \frac{1}{18}Z(s)g(X,Y) \tag{4.112}$$

for $X,Y,Z \in T_p\mathbf{M}, p \in \mathbf{M}$. Using (4.112), it is easy to verify that (3.11) is satisfied. This implies that the Jacobi operator and its covariant derivative commute. Hence, (**M**,g) is a $\mathcal{P}$-space by the results in [4]. **Q.E.D.**

To complete the proof of Theorem 1.2 we have to pick out those of the spaces $I)-IX)$ of Theorem 1.1 which are $\mathcal{Q}$-spaces. We keep all notations from the previous section. Let we set

$$H(X,Y,Z) = \frac{2}{9}X(s)g(Y,Z) + \frac{1}{18}Y(s)g(X,Z) + \frac{1}{18}Z(s)g(X,Y)$$

for $X,Y,Z \in T_p\mathbf{M}$, $p \in \mathbf{M}$. If $(\mathbf{M},g)$ is a space of any of the types $I)-IX)$ then choosing $p \in \Omega$, U_p and E_i, $i = 1,2,3,4$ as in the previous section, we obtain that (4.112) is equivalent to

$$(\nabla_i\rho)_{ii} - H_{iii} = 0, \tag{4.113}$$

$$(\nabla_i\rho)_{jj} - H_{ijj} = 0, \tag{4.114}$$

$$(\nabla_i\rho)_{ij} - H_{iij} = 0, \tag{4.115}$$

$$(\nabla_i\rho)_{jk} - H_{ijk} = 0. \tag{4.116}$$

If the Ricci tensor has not an eigenvalue of the multiplicity three on U_p, then the formulas (3.9) and (3.10) are equivalent to

$$E_i(r_i) = \frac{3}{2}E_i(r_k), \tag{4.117}$$

$$E_i(r_k) = \frac{1}{4}(\nabla_k\rho)_{ki}. \tag{4.118}$$

The equalities (4.117) and (4.118) together with (3.6) and (3.8) imply (4.113), (4.114), (4.115)and (4.116). Thus, any locally conformal flat $\mathcal{P}$-space which is not of type III_1 is an $\mathcal{Q}$-space since its Ricci tensor has not three equal eigenvalues.

Let (**M**,g) be of type III_1). Then the Ricci tensor has an eigenvalue of multiplicity three, say $r_1 = r_2 = r_3 \neq r_4$. The formulas (4.117) and (4.118) are valid for $i \neq 4$. The conditions (4.113), (4.114), (4.115) and (4.116) are equivalent to

$$(\nabla_4\rho)_{44} - H_{444} = 0, \quad (\nabla_4\rho)_{4j} - H_{44j} = 0, \quad (\nabla_4\rho)_{jj} - H_{4jj} = 0, \quad j \in \{1,2,3\}.$$

Using (3.6) (3.8) and (3.9) it is easy to see that the latter three equalities are equivalent to the relation

$$E_4(r_4) = \frac{3}{2}E_4(r_1).$$

It follows from the latter equality an (3.20) that the function F=1/f is a solution of the differential equation (1.2. This completes the proof of Theorem 1.2. **Q.E.D.**

To prove the Remarks, we shall use the fact that a locally conformal flat Riemannian 4-manifold has parallel Ricci tensor iff it has constant Ricci eigenvalues [26, 20].

For Remark 2, it follows from (3.20) that $\mathbf{B^1} \times_{\mathbf{f}} \mathbf{N^3}$ has constant Ricci eigenvalues iff the function $\mathbf{f}$ on $\mathbf{B^1}$ is given by a), b) and c). In this case (1.2) is also satisfied. Remark 2 is contained also in [10].

For Remark 3, it follows from (3.25), (3.40) and (3.33) that the Ricci eigenvalues $r_1 = r_2 \neq r_3 \neq r_4 \neq r_1$ are all constant iff the function f is a constant and $\mathbf{B^2}$ is of constant sectional curvature.

Remark 4 follows immediately from the formulas (3.102), (3.105), (3.108) and (3.111).

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